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ON COUPLED MEAN FIELDS IN HETEROGENEOUS LINEAR THERMOELASTIC MEDIA

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Abstract—The governing equation of the coupled mean fields induced in the heterogeneous linear thermoelastic media was derived. Discussion was made on the nonlocality of the governing equation. The relation of the mean fields was investigated in the uncoupled case when the acceleration term was disregarded.

1. INTRODUCTION

This study is devoted to a statistical formulation of the coupled mean fields induced in the heterogeneous linear thermoelastic media.

Many statistical researches have been conducted on the heterogeneous elastic material. The mean fields induced in the material and the effective elastic moduli tensor of the material were fully investigated by means of the self-consistent method[1–3] or the perturbation method[4–7]. The bounds for the elastic moduli tensor were also examined with the variational principles[8–10]. The results of these investigations made a great contribution to the study on the mechanical behavior of the polycrystalline materials, the composite materials and the multi-phase materials, etc. The investigation from the statistical point of view was also carried out in such fields as the electromagnetism, the dielectric elasticity, the plasticity and the turbulence problems, etc. The works up to this time on the statistical continuum mechanics were summarized in the review articles by Hashin[11] and Beran[12] and the monographs by Beran[13] and Kröner[14].

Beran and McCoy[5, 15] derived a governing equation of the mean field induced in the heterogeneous elastic material in terms of Green's function. They revealed in the paper that the mean field was governed by the nonlocal equation though each constituent had the local property. They also indicated that their statistical formulation had a close resemblance to a linear version of the first strain-gradient theory[16].

In this paper we applied their method of analysis to the thermomechanical study on the heterogeneous linear thermoelastic material. In Section 2 the deterministic formulation was briefly discussed on the coupled linear thermoelastic problem in terms of Green's function method. Section 3 was devoted to the derivation of the governing equation of the mean fields induced in the material. In the derivation, the result obtained in Section 2 was fully used. The reduction to the uncoupled case was examined in Section 4. The relation of the mean fields was derived when the acceleration term was disregarded.

2. FORMULATION OF DETERMINISTIC COUPLED THERMOELASTIC PROBLEM

In this section, we make a brief discussion on a deterministic formulation of the coupled thermoelastic analysis by means of Green's function method which was proposed by the same authors[17]. The result will fully be used in the statistical formulation in the next section.

The field equation of the coupled linear thermoelastic problem is written as[18, 19],

$$\begin{aligned} \rho\ddot{\mathbf{u}} + \operatorname{div}[\mathbf{C}:\operatorname{grad}\mathbf{u} + \boldsymbol{\Theta}\theta] &= -\mathbf{F} \\ \boldsymbol{\lambda}:\operatorname{grad}\operatorname{grad}\theta + c\dot{\theta} &= -\boldsymbol{\Theta}:\operatorname{grad}\dot{\mathbf{u}} - H, \end{aligned} \tag{2.1}$$

where $\mathbf{u}$ and θ represent the displacement and the temperature, respectively. The material constants ρ, $\mathbf{C}$ and $\boldsymbol{\lambda}$ mean the density, the elastic moduli tensor and the heat conductivity tensor, respectively, while c and $\boldsymbol{\Theta}$ are the material constants which relate to the specific heat and the coefficient of thermal expansion, respectively. The functions $\mathbf{F}$ and H stand for the body force and the heat source term which are the prescribed functions of position and time. The operators div and grad show the divergence and the gradient with respect to the spatial coordinate, respectively. The colon indicates the summation over the repeated two indices, i.e. $[\mathbf{A}:\mathbf{B}]_{\ldots} \equiv A_{\ldots ij} B_{ij\ldots}$ The superimposed dot denotes the material time derivative, which is equal to the partial time derivative in this study because the linear theory is treated.

We assume that the material constants ρ, $\mathbf{C}$, $\boldsymbol{\Theta}$, $\boldsymbol{\lambda}$ and c are independent on both time and position. The discussion is limited to the infinite thermoelastic material subjected to the homogeneous boundary conditions. This is the reason why no mention is made on the boundary conditions in this section.

The solutions of eqn (2.1) are formally obtained by means of Green's function method. The results are

$$\mathbf{u}(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}_1} \mathbf{K}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot[\mathrm{div}^1 \{\boldsymbol{\Theta}\theta(\boldsymbol{\xi}_1)\} + \mathbf{F}(\boldsymbol{\xi}_1)]\, \mathrm{d}\boldsymbol{\xi}_1 \tag{2.2}$$

and

$$\theta(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}_1} G(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\, [\boldsymbol{\Theta}: \mathrm{grad}^1\, \dot{\mathbf{u}}(\boldsymbol{\xi}_1) + H(\boldsymbol{\xi}_1)]\, \mathrm{d}\boldsymbol{\xi}_1, \tag{2.3}$$

where Green's functions $\mathbf{K}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ and $G(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ are the solutions of the following equations:

$$\rho\ddot{\mathbf{K}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) + \mathrm{div}\, [\mathbf{C}: \mathrm{grad}\, \mathbf{K}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)] = -\mathbf{I}\, \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_1) \tag{2.4}$$

and

$$\boldsymbol{\lambda}: \mathrm{grad\, grad}\, G(\boldsymbol{\xi}, \boldsymbol{\xi}_1) + c\dot{G}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) = -\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_1). \tag{2.5}$$

In the derivation, the abbreviations $\boldsymbol{\xi}_i \equiv (\mathbf{x}_i, t_i)$ and $d\boldsymbol{\xi}_i \equiv d\mathbf{x}_i dt_i$; ($i$: no sum) are used, where $d\mathbf{x}_i$ means the element of volume. The notations div^i and grad^i mean the divergence and the gradient operators with respect to the spatial coordinate $\mathbf{x}_i$, respectively. Throughout this study, we understand that the time derivative of the two-point function is evaluated with respect to the time constituting the first argument. The integrals are estimated over the region $\mathbf{E}^3 \times \mathbf{R}$, where $\mathbf{E}^3$ and $\mathbf{R}$ mean the three-dimensional infinite space and $[0, \infty)$, respectively. The function $\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_1)$ shows $\delta(\mathbf{x} - \mathbf{x}_1)\, \delta(t - t_1)$, where $\delta(\cdot)$ means the Dirac delta function. The tensor $\mathbf{I}$ stands for the unit tensor. The operator $\cdot$ indicates the inner product, i.e. $[\mathbf{A}\cdot\mathbf{B}]_{\ldots} \equiv A_{\ldots i} B_{i\ldots}$.

It should be noted that if it is necessary, the integral must be estimated in the sense of Cauchy's principal value integral [5, 20, 21].

The problem to seek $\mathbf{K}$ in eqn (2.4) is called the Stokes' problem, and the solution for an infinite isotropic elastic material is available [22]. One can also expect the solution G in eqn (2.5) for an infinite isotropic body [18].

By substituting θ in eqn (2.3) into eqn (2.2) and carrying out the slight manipulation, we obtain the following integral equation in $\boldsymbol{\epsilon}(\boldsymbol{\xi}) \equiv \mathrm{grad}\, \dot{\mathbf{u}}(\boldsymbol{\xi})$:

$$\boldsymbol{\epsilon}(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}_1} \boldsymbol{\alpha}(\boldsymbol{\xi}, \boldsymbol{\xi}_1): \boldsymbol{\epsilon}(\boldsymbol{\xi}_1)\, d\boldsymbol{\xi}_1 + \mathbf{f}(\boldsymbol{\xi}) + \mathbf{h}(\boldsymbol{\xi}). \tag{2.6}$$

The tensor functions $\boldsymbol{\alpha}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$, $\mathbf{f}(\boldsymbol{\xi})$ and $\mathbf{h}(\boldsymbol{\xi})$ are defined as,

$$\begin{aligned} \boldsymbol{\alpha}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) &\equiv \mathrm{grad}\, \dot{\mathbf{A}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \otimes \boldsymbol{\Theta} \\ \mathbf{f}(\boldsymbol{\xi}) &\equiv \int_{\boldsymbol{\xi}_1} \mathrm{grad}\, \dot{\mathbf{K}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot\mathbf{F}(\boldsymbol{\xi}_1)\, \mathrm{d}\boldsymbol{\xi}_1 \\ \mathbf{h}(\boldsymbol{\xi}) &\equiv \int_{\boldsymbol{\xi}_1} \mathrm{grad}\, \dot{\mathbf{A}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\, H(\boldsymbol{\xi}_1)\, \mathrm{d}\boldsymbol{\xi}_1 \end{aligned} \tag{2.7}$$

together with

$$\mathbf{A}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \equiv \int_{\boldsymbol{\xi}_2} \mathbf{K}(\boldsymbol{\xi}, \boldsymbol{\xi}_2)\cdot \mathrm{div}^2 [G(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)\,\boldsymbol{\Theta}]\,\mathrm{d}\boldsymbol{\xi}_2, \tag{2.8}$$

where $\otimes$ represents the tensor product. The solution of eqn (2.6) yields

$$\boldsymbol{\epsilon}(\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\xi}) + \mathbf{h}(\boldsymbol{\xi}) + \int_{\boldsymbol{\xi}_1} \Gamma(\boldsymbol{\xi}, \boldsymbol{\xi}_1): \mathbf{f}(\boldsymbol{\xi}_1) + \mathbf{h}(\boldsymbol{\xi}_1)]\,\mathrm{d}\boldsymbol{\xi}_1, \tag{2.9}$$

where $\Gamma(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ is the resolvent kernel of eqn (2.6)[20, 21]. Equation (2.9) is rewritten as

$$\boldsymbol{\epsilon}(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}_1} [\check{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot \mathbf{F}(\boldsymbol{\xi}_1) + \check{\boldsymbol{\Omega}}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\, H(\boldsymbol{\xi}_1)]\,\mathrm{d}\boldsymbol{\xi}_1. \tag{2.10}$$

To derive eqn (2.10), use is made of eqn $(2.7)_{2,3}$ and the tensors $\check{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ and $\check{\boldsymbol{\Omega}}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ are defined as,

$$\begin{aligned}
\check{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) &\equiv \mathrm{grad}\,\dot{\mathbf{K}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) + \int_{\boldsymbol{\xi}_2} \Gamma(\boldsymbol{\xi}, \boldsymbol{\xi}_2): \mathrm{grad}^2\dot{\mathbf{K}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)\,\mathrm{d}\boldsymbol{\xi}_2 \\
\check{\boldsymbol{\Omega}}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1) &\equiv \mathrm{grad}\,\dot{\mathbf{A}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) + \int_{\boldsymbol{\xi}_2} \Gamma(\boldsymbol{\xi}, \boldsymbol{\xi}_2): \mathrm{grad}^2\dot{\mathbf{A}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)\,\mathrm{d}\boldsymbol{\xi}_2.
\end{aligned} \tag{2.11}$$

Substitution of eqn (2.10) into eqn (2.3) and then of eqn (2.3) into eqn (2.2) gives the final form of the solutions for the coupled linear thermoelastic problem,

$$\mathbf{u}(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}_1} [\boldsymbol{\Omega}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot \mathbf{F}(\boldsymbol{\xi}_1) + \boldsymbol{\Omega}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\, H(\boldsymbol{\xi}_1)]\,\mathrm{d}\boldsymbol{\xi}_1 \tag{2.12}$$

and

$$\theta(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}_1} [\boldsymbol{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot \mathbf{F}(\boldsymbol{\xi}_1) + \Lambda_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\, H(\boldsymbol{\xi}_1)]\,\mathrm{d}\boldsymbol{\xi}_1 \tag{2.13}$$

where we have introduced the following tensor functions:

$$\begin{aligned}
\boldsymbol{\Omega}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) &\equiv \mathbf{K}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) + \int_{\boldsymbol{\xi}_2} [\mathbf{A}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \otimes \boldsymbol{\Theta}]: \check{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)]\,\mathrm{d}\boldsymbol{\xi}_2 \\
\boldsymbol{\Omega}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1) &\equiv \mathbf{A}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) + \int_{\boldsymbol{\xi}_2} [\mathbf{A}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \otimes \boldsymbol{\Theta}]: \check{\boldsymbol{\Omega}}_H(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)]\,\mathrm{d}\boldsymbol{\xi}_2
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
\boldsymbol{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) &\equiv \int_{\boldsymbol{\xi}_2} G(\boldsymbol{\xi}, \boldsymbol{\xi}_2)\,\boldsymbol{\Theta}: \check{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)\,\mathrm{d}\boldsymbol{\xi}_2 \\
\Lambda_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1) &\equiv G(\boldsymbol{\xi}, \boldsymbol{\xi}_2) + \int_{\boldsymbol{\xi}_2} G(\boldsymbol{\xi}, \boldsymbol{\xi}_2)\,\boldsymbol{\Theta}: \check{\boldsymbol{\Omega}}_H(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)\,\mathrm{d}\boldsymbol{\xi}_2.
\end{aligned} \tag{2.15}$$

The problem can be reduced to the uncoupled one if the condition

$$\boldsymbol{\Theta}: \mathrm{grad}\,\dot{\mathbf{u}} = 0 \tag{2.16}$$

holds. We can show that under the condition (2.16), eqns (2.14) and (2.15) are simplified to

$$\boldsymbol{\Omega}_{\mathbf{F}} = \mathbf{K}, \qquad \boldsymbol{\Omega}_H = \mathbf{A}, \qquad \boldsymbol{\Lambda}_{\mathbf{F}} = \mathbf{0}, \qquad \Lambda_H = G. \tag{2.17}$$

3. GOVERNING EQUATION OF COUPLED MEAN FIELDS

The governing equation of the mean fields induced in the infinite heterogeneous thermoelastic material is examined in this section.

The fundamental equation is eqn (2.1). However, in this section, the material constants ρ, $\mathbf{C}$, $\boldsymbol{\Theta}$, $\boldsymbol{\lambda}$ and c are assumed to be the statistically homogeneous random functions of the position, and to have mutually no correlations. For simplicity, we further assume that the body force, the heat source term and the prescribed boundary conditions behave in a non-random fashion.

Now it is postulated that the material constants can be expressed as the summation of the mean value and the deviation about it[5], i.e.

$$\begin{gathered}\rho(\mathbf{x}) = \langle\rho(\mathbf{x})\rangle + \rho'(\mathbf{x}), \qquad \mathbf{C}(\mathbf{x}) = \langle\mathbf{C}(\mathbf{x})\rangle + \mathbf{C}'(\mathbf{x}),\\ \boldsymbol{\Theta}(\mathbf{x}) = \langle\boldsymbol{\Theta}(\mathbf{x})\rangle + \boldsymbol{\Theta}'(\mathbf{x}), \qquad \boldsymbol{\lambda}(\mathbf{x}) = \langle\boldsymbol{\lambda}(\mathbf{x})\rangle + \boldsymbol{\lambda}'(\mathbf{x}),\\ c(\mathbf{x}) = \langle c(\mathbf{x})\rangle + c'(\mathbf{x}),\end{gathered} \tag{3.1}$$

where the notations $\langle\,\rangle$ and $'$ indicate the ensemble average and the deviation from the average, respectively. The assumption of the statistically homogeneity implies the ensemble average to be independent of the position.

The random property of the material constants makes the displacement and the temperature fluctuate randomly. They are, thus, decomposed as

$$\mathbf{u}(\boldsymbol{\xi}) = \langle\mathbf{u}(\boldsymbol{\xi})\rangle + \mathbf{u}'(\boldsymbol{\xi}), \qquad \theta(\boldsymbol{\xi}) = \langle\theta(\boldsymbol{\xi})\rangle + \theta'(\boldsymbol{\xi}). \tag{3.2}$$

It should be noted that $\mathbf{u}$ and θ are the functions of both the position and the time, and that mean values $\langle\mathbf{u}(\boldsymbol{\xi})\rangle$ and $\langle\theta(\boldsymbol{\xi})\rangle$ are not constants.

In order to get the governing equation of $\langle\mathbf{u}\rangle$ and $\langle\theta\rangle$, we substitute eqns (3.1) and (3.2) into eqn (2.1) and take average. The result is

$$\begin{gathered}\langle\rho\rangle\langle\ddot{\mathbf{u}}\rangle + \operatorname{div}\,[\langle\mathbf{C}\rangle\colon \operatorname{grad}\langle\mathbf{u}\rangle + \langle\boldsymbol{\Theta}\rangle\langle\theta\rangle] + \langle\rho'\ddot{\mathbf{u}}'\rangle + \operatorname{div}\,[\langle\mathbf{C}'\colon \operatorname{grad}\mathbf{u}'\rangle + \langle\boldsymbol{\Theta}'\theta'\rangle] = -\mathbf{F},\\ \langle\boldsymbol{\lambda}\rangle\colon \operatorname{grad}\operatorname{grad}\langle\theta\rangle + \langle c\rangle\langle\dot{\theta}\rangle + \langle\boldsymbol{\Theta}\rangle\colon \operatorname{grad}\langle\dot{\mathbf{u}}\rangle + \langle\boldsymbol{\lambda}'\colon \operatorname{grad}\operatorname{grad}\theta'\rangle + \langle c'\dot{\theta}'\rangle + \langle\boldsymbol{\Theta}'\colon \operatorname{grad}\dot{\mathbf{u}}'\rangle = -H.\end{gathered} \tag{3.3}$$

The equation for the deviations is lead from eqn (3.3) and (2.1) to

$$\begin{gathered}\langle\rho\rangle\ddot{\mathbf{u}}' + \rho'\langle\ddot{\mathbf{u}}\rangle + (I-P)(\rho'\ddot{\mathbf{u}}') + \operatorname{div}\,[\langle\mathbf{C}\rangle\colon \operatorname{grad}\mathbf{u}' + \mathbf{C}'\colon \operatorname{grad}\langle\mathbf{u}\rangle + (I-P)(\mathbf{C}'\colon \operatorname{grad}\mathbf{u}')\\ + \langle\boldsymbol{\Theta}\rangle\theta' + \boldsymbol{\Theta}'\langle\theta\rangle + (I-P)(\boldsymbol{\Theta}'\theta')] = \mathbf{0},\\ \langle\boldsymbol{\lambda}\rangle\colon \operatorname{grad}\operatorname{grad}\theta' + \boldsymbol{\lambda}'\colon \operatorname{grad}\operatorname{grad}\langle\theta\rangle + (I-P)(\boldsymbol{\lambda}'\colon \operatorname{grad}\operatorname{grad}\theta') + \langle c\rangle\dot{\theta}' + c'\langle\dot{\theta}\rangle\\ + (I-P)(c'\dot{\theta}') + \langle\boldsymbol{\Theta}\rangle\colon \operatorname{grad}\dot{\mathbf{u}}' + \boldsymbol{\Theta}'\colon \operatorname{grad}\langle\dot{\mathbf{u}}\rangle + (I-P)(\boldsymbol{\Theta}'\colon \operatorname{grad}\dot{\mathbf{u}}') = 0,\end{gathered} \tag{3.4}$$

where we have used the operators I and P defined as[5, 14] $I(\mathbf{A}) \equiv \mathbf{A}, P(\mathbf{A}) \equiv \langle\mathbf{A}\rangle$.

To complete eqn (3.3) for the mean fields $\langle\mathbf{u}\rangle$ and $\langle\theta\rangle$, one must express the terms which include the deviations in terms of $\langle\mathbf{u}\rangle$ and $\langle\theta\rangle$. For that purpose, we first solve eqn (3.4) in $\mathbf{u}'$ and θ' by assuming that the iteration procedure is applicable to this problem. We set

$$\mathbf{u}'(\boldsymbol{\xi}) = \sum_{n=1}^{\infty} \mathbf{u}'^{(n)}(\boldsymbol{\xi}), \qquad \theta'(\boldsymbol{\xi}) = \sum_{n=1}^{\infty} \theta'^{(n)}(\boldsymbol{\xi}) \tag{3.5}$$

with the postulation of the convergence of the series. Equation (4.3) is satisfied if the following equations hold for $\mathbf{u}'^{(n)}(\boldsymbol{\xi})$ and $\theta'^{(n)}(\boldsymbol{\xi})$:

$$\begin{gathered}\langle\rho\rangle\ddot{\mathbf{u}}'^{(1)} + \operatorname{div}\,[\langle\mathbf{C}\rangle\colon \operatorname{grad}\mathbf{u}'^{(1)} + \langle\boldsymbol{\Theta}\rangle\theta'^{(1)}] = -\operatorname{div}\,[\mathbf{C}'\colon \operatorname{grad}\langle\mathbf{u}\rangle + \boldsymbol{\Theta}'\langle\theta\rangle] - \rho'\langle\ddot{\mathbf{u}}\rangle,\\ \langle\rho\rangle\ddot{\mathbf{u}}'^{(n)} + \operatorname{div}\,[\langle\mathbf{C}\rangle\colon \operatorname{grad}\mathbf{u}'^{(n)} + \langle\boldsymbol{\Theta}\rangle\theta'^{(n)}]\\ = -(I-P)\,[\operatorname{div}\,(\mathbf{C}'\colon \operatorname{grad}\mathbf{u}'^{(n-1)} + \boldsymbol{\Theta}'\theta'^{(n-1)}) + \rho'\ddot{\mathbf{u}}'^{(n-1)}];\ (n > 1),\end{gathered} \tag{3.6}$$

and

$$\langle\boldsymbol{\lambda}\rangle\colon \operatorname{grad}\operatorname{grad}\theta'^{(1)} + \langle c\rangle\dot{\theta}'^{(1)} + \langle\boldsymbol{\Theta}\rangle\colon \operatorname{grad}\dot{\mathbf{u}}'^{(1)} = -c'\langle\dot{\theta}\rangle - \boldsymbol{\lambda}'\colon \operatorname{grad}\operatorname{grad}\langle\theta\rangle - \boldsymbol{\Theta}'\colon \operatorname{grad}\langle\dot{\mathbf{u}}\rangle,$$

$$\langle \boldsymbol{\lambda} \rangle : \operatorname{grad}\operatorname{grad}\theta'^{(n)} + \langle c \rangle \dot{\theta}'^{(n)} + \langle \boldsymbol{\Theta} \rangle : \operatorname{grad}\dot{\mathbf{u}}'^{(n)} = -(I-P)[\boldsymbol{\lambda}' : \operatorname{grad}\operatorname{grad}\theta'^{(n-1)} + c'\,\dot{\theta}'^{(n-1)} + \boldsymbol{\Theta}' : \operatorname{grad}\dot{\mathbf{u}}'^{(n-1)}];\ (n>1). \tag{3.7}$$

It is easy to give the solutions in eqns (3.6) and (3.7), if we make use of the result in the preceding section. For example, $\mathbf{u}'^{(1)}(\boldsymbol{\xi})$ and $\theta'^{(1)}(\boldsymbol{\xi})$ in eqns $(3.6)_1$ and $(3.7)_1$ are obtained by taking the following replacement in eqn (2.1):

$$\rho \to \langle \rho \rangle,\ \mathbf{C} \to \langle \mathbf{C} \rangle,\ \boldsymbol{\Theta} \to \langle \boldsymbol{\Theta} \rangle,\ \boldsymbol{\lambda} \to \langle \boldsymbol{\lambda} \rangle,\ c \to \langle c \rangle, \tag{3.8}$$

and

$$\mathbf{F} \to \operatorname{div}[\mathbf{C}' : \operatorname{grad}\langle \mathbf{u} \rangle + \boldsymbol{\Theta}'\langle \theta \rangle] + \rho'\langle \ddot{\mathbf{u}} \rangle$$
$$H \to c'\langle \dot{\theta} \rangle + \boldsymbol{\lambda}' : \operatorname{grad}\operatorname{grad}\langle \theta \rangle + \boldsymbol{\Theta}' : \operatorname{grad}\langle \dot{\mathbf{u}} \rangle. \tag{3.9}$$

The same procedure can be applied in seeking the solutions $\mathbf{u}'^{(n)}(\boldsymbol{\xi})$, $\theta'^{(n)}(\boldsymbol{\xi})$; $(n>1)$. The results read,

$$\mathbf{u}'^{(1)}(\boldsymbol{\xi}) = \int_{\xi_1} [\![\boldsymbol{\Omega}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot[\operatorname{div}^1\{\mathbf{C}'(\mathbf{x}_1) : \operatorname{grad}^1\langle \mathbf{u}(\boldsymbol{\xi}_1)\rangle + \boldsymbol{\Theta}'(\mathbf{x}_1)\langle \theta(\boldsymbol{\xi}_1)\rangle\} + \rho'(\mathbf{x}_1)\langle \ddot{\mathbf{u}}(\boldsymbol{\xi}_1)\rangle]$$
$$+ \boldsymbol{\Omega}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)[\boldsymbol{\lambda}'(\mathbf{x}_1) : \operatorname{grad}^1\operatorname{grad}^1\langle \theta(\boldsymbol{\xi}_1)\rangle + c'(\mathbf{x}_1)\langle \dot{\theta}(\boldsymbol{\xi}_1)\rangle + \boldsymbol{\Theta}'(\mathbf{x}_1) : \operatorname{grad}^1\langle \dot{\mathbf{u}}(\boldsymbol{\xi}_1)\rangle]]\!]\, d\boldsymbol{\xi}_1,$$

$$\mathbf{u}'^{(n)}(\boldsymbol{\xi}) = (I-P)\int_{\xi_1} [\![\boldsymbol{\Omega}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot[\operatorname{div}^1\{\mathbf{C}'(\mathbf{x}_1) : \operatorname{grad}^1\mathbf{u}'^{(n-1)}(\boldsymbol{\xi}_1) + \boldsymbol{\Theta}'(\mathbf{x}_1)\theta'^{(n-1)}(\boldsymbol{\xi}_1)\} + \rho'(\mathbf{x}_1)\ddot{\mathbf{u}}'^{(n-1)}(\boldsymbol{\xi}_1)]$$
$$+ \boldsymbol{\Omega}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)[\boldsymbol{\lambda}'(\mathbf{x}_1) : \operatorname{grad}^1\operatorname{grad}^1\theta'^{(n-1)}(\boldsymbol{\xi}_1)$$
$$+ c'(\mathbf{x}_1)\,\dot{\theta}'^{(n-1)}(\boldsymbol{\xi}_1) + \boldsymbol{\Theta}'(\mathbf{x}_1) : \operatorname{grad}^1\dot{\mathbf{u}}'^{(n-1)}(\boldsymbol{\xi}_1)]]\!]\, d\boldsymbol{\xi}_1;\ (n>1),$$

and

$$\theta'^{(1)}(\boldsymbol{\xi}) = \int_{\xi_1} [\![\boldsymbol{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot[\operatorname{div}^1\{\mathbf{C}'(\mathbf{x}_1) : \operatorname{grad}^1\langle \mathbf{u}(\boldsymbol{\xi}_1)\rangle + \boldsymbol{\Theta}'(\mathbf{x}_1)\langle \theta(\boldsymbol{\xi}_1)\rangle\} + \rho'(\mathbf{x}_1)\langle \ddot{\mathbf{u}}(\boldsymbol{\xi}_1)\rangle]$$
$$+ \Lambda_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)[\boldsymbol{\lambda}'(\mathbf{x}_1) : \operatorname{grad}^1\operatorname{grad}^1\langle \theta(\boldsymbol{\xi}_1)\rangle + c'(\mathbf{x}_1)\langle \dot{\theta}(\boldsymbol{\xi}_1)\rangle + \boldsymbol{\Theta}'(\mathbf{x}_1) : \operatorname{grad}^1\langle \dot{\mathbf{u}}(\boldsymbol{\xi}_1)\rangle]]\!]\, d\boldsymbol{\xi}_1,$$

$$\theta'^{(n)}(\boldsymbol{\xi}) = (I-P)\int_{\xi_1} [\![\boldsymbol{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot[\operatorname{div}^1\{\mathbf{C}'(\mathbf{x}_1) : \operatorname{grad}^1\mathbf{u}'^{(n-1)}(\boldsymbol{\xi}_1) + \boldsymbol{\Theta}'(\mathbf{x}_1)\,\theta'^{(n-1)}(\boldsymbol{\xi}_1)\} + \rho'(\mathbf{x}_1)\,\ddot{\mathbf{u}}'^{(n-1)}(\boldsymbol{\xi}_1)]$$
$$+ \Lambda_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)[\boldsymbol{\lambda}'(\mathbf{x}_1) : \operatorname{grad}^1\theta'^{(n-1)}(\boldsymbol{\xi}_1)$$
$$+ c'(\mathbf{x}_1)\,\dot{\theta}'^{(n-1)}(\boldsymbol{\xi}_1) + \boldsymbol{\Theta}'(\mathbf{x}_1) : \operatorname{grad}^1\dot{\mathbf{u}}'^{(n-1)}(\boldsymbol{\xi}_1)]]\!]\, d\boldsymbol{\xi}_1;\ (n>1), \tag{3.11}$$

where the tensors $\boldsymbol{\Omega}_{\mathbf{F}}$, $\boldsymbol{\Omega}_{\mathbf{H}}$, $\boldsymbol{\Lambda}_{\mathbf{F}}$ and Λ_H are defined by eqns (2.14) and (2.15). It should be noted that in this case the material constants are replaced by the mean values as indicated in eqn (3.8). Green's functions $\mathbf{K}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ and $G(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ are, therefore, determined for the body with the material constants $\langle \rho \rangle$, $\langle \mathbf{C} \rangle$, etc.

Now we can estimate $\langle \rho'\ddot{\mathbf{u}}' \rangle$, $\langle \mathbf{C}' : \operatorname{grad}\mathbf{u}' \rangle$, $\langle \boldsymbol{\Theta}'\theta' \rangle$, $\langle \boldsymbol{\lambda}' : \operatorname{grad}\operatorname{grad}\theta' \rangle$, $\langle c'\dot{\theta}' \rangle$ and $\langle \boldsymbol{\Theta}' : \operatorname{grad}\dot{\mathbf{u}}' \rangle$ by making use of the deviations $\mathbf{u}'^{(n)}$ and $\theta'^{(n)}$ in eqns (3.10) and (3.11). Equations $(3.5)_1$ leads to

$$\langle \rho'\ddot{\mathbf{u}}' \rangle = {}_I\boldsymbol{\chi}(\boldsymbol{\xi}), \tag{3.12}$$

where the functionals ${}_K\boldsymbol{\chi}(\boldsymbol{\xi})$; $(K = I, II \ldots VI)$ are defined by

$${}_K\boldsymbol{\chi}(\boldsymbol{\xi}) \equiv \int_{\Sigma_1} {}_K\mathfrak{A}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot\langle \ddot{\mathbf{u}}(\boldsymbol{\xi}_1)\rangle\, d\Sigma_1 + \int_{\xi_1} {}_K\mathfrak{A}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\cdot\ddot{\mathbf{u}}(\boldsymbol{\xi}_1)\rangle\, d\boldsymbol{\xi}_1$$
$$+ \int_{\Sigma_1} {}_K\mathfrak{B}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1) : \operatorname{grad}^1\langle \mathbf{u}(\boldsymbol{\xi}_1)\rangle\, d\Sigma_1 + \int_{\xi_1} {}_K\mathfrak{B}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1) : \operatorname{grad}^1\langle \mathbf{u}(\boldsymbol{\xi}_1)\rangle\, d\boldsymbol{\xi}_1$$
$$+ \int_{\Sigma_1} {}_K\mathfrak{C}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\langle \theta(\boldsymbol{\xi}_1)\rangle\, d\Sigma_1 + \int_{\Sigma_1} {}_K\mathfrak{C}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\langle \theta(\boldsymbol{\xi}_1)\rangle\, d\boldsymbol{\xi}_1$$

$$+\int_{\Sigma_1} {}_K\mathfrak{D}_1(\boldsymbol{\xi},\boldsymbol{\xi}_1):\mathrm{grad}^1\,\mathrm{grad}^1\langle\theta(\boldsymbol{\xi}_1)\rangle\,\mathrm{d}\Sigma_1+\int_{\boldsymbol{\xi}_1} {}_K\mathfrak{D}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1):\mathrm{grad}^1\,\mathrm{grad}^1\langle\theta(\boldsymbol{\xi}_1)\rangle\,\mathrm{d}\boldsymbol{\xi}_1$$
$$+\int_{\Sigma_1} {}_K\mathfrak{E}_1(\boldsymbol{\xi},\boldsymbol{\xi}_1)\langle\dot\theta(\boldsymbol{\xi}_1)\rangle\,\mathrm{d}\Sigma_1+\int_{\boldsymbol{\xi}_1} {}_K\mathfrak{E}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1)\langle\dot\theta(\boldsymbol{\xi}_1)\rangle\,\mathrm{d}\boldsymbol{\xi}_1+\int_{\Sigma_1} {}_K\mathfrak{F}_1(\boldsymbol{\xi},\boldsymbol{\xi}_1):\mathrm{grad}^1\langle\dot{\mathbf{u}}(\boldsymbol{\xi}_1)\rangle\,\mathrm{d}\Sigma_1$$
$$+\int_{\boldsymbol{\xi}_1} {}_K\mathfrak{F}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1):\mathrm{grad}^1\langle\dot{\mathbf{u}}(\boldsymbol{\xi}_1)\rangle\,\mathrm{d}\boldsymbol{\xi}_1 \tag{3.13}$$

The functionals ${}_I\chi$, ${}_K\chi$; ($K = II, III$) and ${}_K\chi$; (K = IV, V, VI) are the tensors of the first, the second and the zeroth rank, respectively. The functions ${}_K\mathfrak{A}_1, \ldots$ etc., which have the corresponding ranks estimated from eqn (3.13), will be defined later. They are determined by Green's functions and the infinite set of the correlation functions of the material constants. Their explicit forms are so complicated that we write down in this paper the only ones which contain the correlation functions of the same material constant in first two terms. Henceforth, we call these functions as the primary functions.

The tensor function ${}_I\mathfrak{A}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1)$ is the only primary function in eqn (3.12). Its first few terms are as follows:

$${}_I\mathfrak{A}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1)\equiv[\![\ddot{\boldsymbol{\Omega}}_F(\boldsymbol{\xi},\boldsymbol{\xi}_1)\,\boldsymbol{\Psi}_\rho^{(2)}(\mathbf{x},\mathbf{x}_1)]\!]+\left[\!\!\left[\int_{\boldsymbol{\xi}_2}\ddot{\boldsymbol{\Omega}}_F(\boldsymbol{\xi},\boldsymbol{\xi}_2)\cdot\ddot{\boldsymbol{\Omega}}_F(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1)\,\boldsymbol{\Psi}_\rho^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)\,\mathrm{d}\boldsymbol{\xi}_2\right]\!\!\right]+\cdots \tag{3.14}$$

Here, we have introduced the following notation for the correlation functions:

$$\boldsymbol{\Psi}_\Delta^{(n)}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\equiv\langle\Delta'(\mathbf{x}_1)\otimes\Delta'(\mathbf{x}_2)\otimes\ldots\otimes\Delta'(\mathbf{x}_n)\rangle, \tag{3.15}$$

where Δ represents the one in the set $(\rho, \mathbf{C}, \boldsymbol{\Theta}, \boldsymbol{\lambda}, c)$.

The term $\langle \mathbf{C}'\!:\,\mathrm{grad}\,\mathbf{u}'\rangle$ can be calculated by substituting eqn (3.10) into

$$\langle\mathbf{C}':\mathrm{grad}\,\mathbf{u}'\rangle=\sum_{n=1}^{\infty}\langle\mathbf{C}':\mathrm{grad}\,\mathbf{u}'^{(n)}\rangle$$

and carrying out a lengthy manipulation;

$$\langle\mathbf{C}':\mathrm{grad}\,\mathbf{u}'\rangle\equiv{}_{II}\chi(\boldsymbol{\xi}), \tag{3.16}$$

where the primary functions in ${}_{II}\chi(\boldsymbol{\xi})$ are

$$\begin{aligned}{}_{II}\mathfrak{B}_1(\boldsymbol{\xi},\boldsymbol{\xi}_1)\equiv{}&[\![\mathrm{grad}\,\boldsymbol{\Omega}_F(\boldsymbol{\xi},\boldsymbol{\xi}_1)\otimes\mathbf{n}(\boldsymbol{\sigma}_1))\boldsymbol{\Delta}\boldsymbol{\Psi}_C^{(2)}(\mathbf{x},\mathbf{x}_1)]\!]\\&+\Big[\!\!\Big[\int_{\Sigma_2}[\mathrm{grad}\,\boldsymbol{\Omega}_F(\boldsymbol{\xi},\boldsymbol{\xi}_2)\otimes\mathbf{n}(\boldsymbol{\sigma}_2)]\boldsymbol{\Delta}[(\mathrm{grad}^2\,\boldsymbol{\Omega}_F(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1)\\&\otimes\mathbf{n}(\boldsymbol{\sigma}_1))\,\boldsymbol{\Delta}\,\boldsymbol{\Psi}_C^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\Sigma_2-\int_{\boldsymbol{\xi}_2}\mathrm{grad}^2\,\mathrm{grad}\,\boldsymbol{\Omega}_F(\boldsymbol{\xi},\boldsymbol{\xi}_2)\\&\boldsymbol{\Delta}\,[(\mathrm{grad}^2\,\boldsymbol{\Omega}_F(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1)\otimes\mathbf{n}(\boldsymbol{\sigma}_1))\,\boldsymbol{\Delta}\,\boldsymbol{\Psi}_C^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\boldsymbol{\xi}_2\Big]\!\!\Big]+\cdots\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}{}_{II}\mathfrak{B}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1)\equiv{}&[\![(\mathrm{grad}^1\,\mathrm{grad}\,\boldsymbol{\Omega}_F(\boldsymbol{\xi},\boldsymbol{\xi}_1)\,\boldsymbol{\Delta}\,\boldsymbol{\Psi}_C^{(2)}(\mathbf{x},\mathbf{x}_1)]\!]+\Big[\!\!\Big[-\int_{\Sigma_2}(\mathrm{grad}\,\boldsymbol{\Omega}_F(\boldsymbol{\xi},\boldsymbol{\xi}_2)\otimes\mathbf{n}(\boldsymbol{\sigma}_2))\\&\boldsymbol{\Delta}\;[\mathrm{grad}^1\,\mathrm{grad}^2\,\boldsymbol{\Omega}_F(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1)\,\boldsymbol{\Delta}\,\boldsymbol{\Psi}_C^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\Sigma_2\\&+\int_{\boldsymbol{\xi}_2}\mathrm{grad}^2\,\mathrm{grad}\,\boldsymbol{\Omega}_F(\boldsymbol{\xi},\boldsymbol{\xi}_2)\,\boldsymbol{\Delta}\,[\mathrm{grad}^1\,\mathrm{grad}^2\,\boldsymbol{\Omega}_F(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1)\,\boldsymbol{\Delta}\,\boldsymbol{\Psi}_C^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\boldsymbol{\xi}_2\Big]\!\!\Big]+\cdots\end{aligned} \tag{3.18}$$

The notation $\boldsymbol{\sigma}_i$ in the above equations means the spatial coordinate $\mathbf{x}_i$ on the boundary surface. The vector $\mathbf{n}(\boldsymbol{\sigma}_i)$ stands for the outward unit normal at the boundary point with the coordinate $\boldsymbol{\sigma}_i$. The abbreviations $\boldsymbol{\Sigma}_i\equiv(\boldsymbol{\sigma}_i, t_i)$ and $d\boldsymbol{\Sigma}_i\equiv d\boldsymbol{\sigma}_i\,dt_i$; ($i$: no sum) are adopted, where $d\boldsymbol{\sigma}_i$ means the element of surface. The operator $\mathbf{\Delta}$ represents the summation over the repeated four indices, i.e. $[\mathbf{A}\,\mathbf{\Delta}\,\mathbf{B}]_{\ldots mn} \equiv A_{\ldots ijkl}\, B_{\ldots ikjlmn}$.

As in the case of the volume integral, the surface integral induced by Green's theorem should be evaluated in the sense of Cauchy's principal value integral[5, 20, 21, 23].

The derivation of eqns (3.16), (3.17) and (3.18) are somewhat complicated. In Appendix, we illustrate the process to calculate $\langle \mathbf{C}' \colon \operatorname{grad} \mathbf{u}'^{(2)} \rangle$.

We can also estimate $\langle \mathbf{\Theta}'\, \theta' \rangle$, $\langle \boldsymbol{\lambda}' \colon \operatorname{grad}\operatorname{grad} \theta' \rangle$, $\langle c' \dot{\theta}' \rangle$ and $\langle \mathbf{\Theta}' \colon \operatorname{grad} \dot{\mathbf{u}}' \rangle$ as follows. It follows from eqns (3.5) and (3.11) that

$$\langle \mathbf{\Theta}'\theta' \rangle = {}_{III}\boldsymbol{\chi}(\boldsymbol{\xi}). \tag{3.19}$$

The primary functions in ${}_{III}\boldsymbol{\chi}(\boldsymbol{\xi})$ are the tensor functions of the second rank ${}_{III}\mathfrak{C}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ and ${}_{III}\mathfrak{C}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1),\boldsymbol{\xi}_1)$ and of the fourth ${}_{III}\mathfrak{F}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1), \boldsymbol{\xi}_1)$,which are determined as

$$\begin{aligned}
{}_{III}\mathfrak{C}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \equiv{}& [\![(\mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_1)) \colon \mathbf{\Psi}^{(2)}_{\mathbf{\Theta}}(\mathbf{x}, \mathbf{x}_1)]\!] \\
&+ \Bigg[\!\!\Bigg[\int_{\Sigma_2} [(\mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \otimes \mathbf{n}(\boldsymbol{\sigma}_2)) \otimes (\mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_2))]\, \mathbf{\Delta}\, \mathbf{\Psi}^{(3)}_{\mathbf{\Theta}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x})\, \mathrm{d}\Sigma_2 \\
&- \int_{\boldsymbol{\xi}_2} [\operatorname{grad}^2 \mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \otimes (\mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_2))\mathbf{\Delta}\mathbf{\Psi}^{(3)}_{\mathbf{\Theta}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x})]\, \mathrm{d}\boldsymbol{\xi}_2 \\
&+ \int_{\boldsymbol{\xi}_2} \mathbf{\Lambda}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_2) [\operatorname{grad}^2 \dot{\mathbf{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_1)]\, \mathbf{\Delta}\mathbf{\Psi}^{(3)}_{\mathbf{\Theta}}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)\, \mathrm{d}\boldsymbol{\xi}_2 \Bigg]\!\!\Bigg] + \cdots,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
{}_{III}\mathfrak{C}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \equiv{}& [\![-\operatorname{grad}^1 \mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \colon \mathbf{\Psi}^{(2)}_{\mathbf{\Theta}}(\mathbf{x}_1, \mathbf{x})]\!] \\
&+ \Bigg\{ - \int_{\Sigma_2} [\mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \otimes \mathbf{n}(\boldsymbol{\sigma}_2)) \otimes \operatorname{grad}^1 \mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)]\, \mathbf{\Delta}\, \mathbf{\Psi}^{(3)}_{\mathbf{\Theta}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x})\, \mathrm{d}\, \Sigma_2 \\
&+ \int_{\boldsymbol{\xi}_2} [\operatorname{grad}^2 \mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \otimes \operatorname{grad}^2 \mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)]\, \mathbf{\Delta}\, \mathbf{\Psi}^{(3)}_{\mathbf{\Theta}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x})]\, \mathrm{d}\boldsymbol{\xi}_2 \\
&- \int_{\boldsymbol{\xi}_2} \mathbf{\Lambda}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \operatorname{grad}^1 \operatorname{grad}^2 \dot{\mathbf{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)\, \mathbf{\Delta}\, \mathbf{\Psi}^{(3)}_{\mathbf{\Theta}}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)\, \mathrm{d}\boldsymbol{\xi}_2 \Bigg]\!\!\Bigg] + \cdots
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
{}_{III}\mathfrak{F}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \equiv{}& [\![\mathbf{\Lambda}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1)\, \mathbf{\Psi}^{(2)}_{\mathbf{\Theta}}(\mathbf{x}, \mathbf{x}_1)]\!] + \Bigg[\!\!\Bigg[\int_{\Sigma_2} \mathbf{\Lambda}_H(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1)\, (\mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \otimes \mathbf{n}(\boldsymbol{\sigma}_2)) \colon \mathbf{\Psi}^{(3)}_{\mathbf{\Theta}}(\mathbf{x}_2, \mathbf{x}, \mathbf{x}_1)\, \mathrm{d}\Sigma_2 \\
&+ \int_{\boldsymbol{\xi}_2} [\mathbf{\Lambda}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \operatorname{grad}^2 \dot{\mathbf{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) - \mathbf{\Lambda}_H(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \operatorname{grad}^2 \mathbf{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_2)] \colon \mathbf{\Psi}^{(3)}_{\mathbf{\Theta}}(\mathbf{x}_2, \mathbf{x}, \mathbf{x}_1)\, \mathrm{d}\boldsymbol{\xi}_2 \Bigg]\!\!\Bigg] + \cdots
\end{aligned}$$

The term $\langle \boldsymbol{\lambda}' \colon \operatorname{grad}\operatorname{grad} \theta' \rangle$ reads

$$\langle \boldsymbol{\lambda}' \colon \operatorname{grad}\operatorname{grad} \theta' \rangle = \sum_{n=1}^{\infty} \langle \boldsymbol{\lambda}' \colon \operatorname{grad}\operatorname{grad} \theta'^{(n)} \rangle = {}_{IV}\boldsymbol{\chi}(\boldsymbol{\xi}), \tag{3.23}$$

where the tensor function of the second rank,

$$\begin{aligned}
{}_{IV}\mathfrak{D}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \equiv{}& [\![\operatorname{grad}\operatorname{grad} \mathbf{\Lambda}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \colon \mathbf{\Psi}^{(2)}_{\boldsymbol{\lambda}}(\mathbf{x}, \mathbf{x}_1)]\!] + \Bigg[\!\!\Bigg[\int_{\boldsymbol{\xi}_2} [\operatorname{grad}^2 \operatorname{grad}^2 \mathbf{\Lambda}_H(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \\
&\otimes \operatorname{grad}\operatorname{grad} \mathbf{\Lambda}_H(\boldsymbol{\xi}, \boldsymbol{\xi}_2)]\, \mathbf{\Delta}\, \mathbf{\Psi}^{(3)}_{\boldsymbol{\lambda}}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)\, \mathrm{d}\boldsymbol{\xi}_2 \Bigg]\!\!\Bigg] + \cdots
\end{aligned}$$

is the only primary function in ${}_{IV}\boldsymbol{\chi}(\boldsymbol{\xi})$.

It also follows that

$$\langle c' \dot{\theta}' \rangle = \sum_{n=1}^{\infty} \langle c' \dot{\theta}'^{(n)} \rangle = {}_{V}\boldsymbol{\chi}(\boldsymbol{\xi}) \tag{3.25}$$

with the primary function

$$ {}_{V}\mathfrak{C}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1) \equiv [\![\dot{\Lambda}_H(\boldsymbol{\xi},\boldsymbol{\xi}_1)\,\boldsymbol{\Psi}_c^{(2)}(\mathbf{x},\mathbf{x}_1)]\!] + \left[\!\!\left[\int_{\boldsymbol{\xi}_2} \dot{\Lambda}_H(\boldsymbol{\xi},\boldsymbol{\xi}_2)\,\dot{\Lambda}_H(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1)\,\boldsymbol{\Psi}_c^{(3)}(\mathbf{x},\mathbf{x}_1,\mathbf{x}_2)\,\mathrm{d}\boldsymbol{\xi}_2 \right]\!\!\right] + \cdots \tag{3.26} $$

and

$$ \langle \boldsymbol{\Theta}' : \operatorname{grad} \dot{\mathbf{u}}' \rangle = \sum_{n=1}^{\infty} \langle \boldsymbol{\Theta}' : \operatorname{grad} \dot{\mathbf{u}}'^{(n)} \rangle = {}_{VI}\chi(\boldsymbol{\xi}) \tag{3.27} $$

together with the primary functions

$$ \begin{aligned} {}_{VI}\mathfrak{C}_1(\boldsymbol{\xi},\boldsymbol{\xi}_1) \equiv {} & [\![(\operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_1)) \blacktriangle \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(2)}(\mathbf{x},\mathbf{x}_1)]\!] \\ & + \left[\!\!\left[\int_{\boldsymbol{\Sigma}_2} (\boldsymbol{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_1)) : [(\operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_2) \otimes \mathbf{n}(\boldsymbol{\sigma}_2)) \,\Delta\, \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\boldsymbol{\Sigma}_2 \right.\right. \\ & - \int_{\boldsymbol{\xi}_2} (\boldsymbol{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_1)) : [\operatorname{grad}^2 \operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_2) \,\Delta\, \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\boldsymbol{\xi}_2 \\ & \left.\left. + \int_{\boldsymbol{\xi}_2} \operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_2) : [(\operatorname{grad}^2 \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) \otimes \boldsymbol{n}(\boldsymbol{\sigma}_1)) \blacktriangle \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\boldsymbol{\xi}_2 \right]\!\!\right] + \ldots, \end{aligned} \tag{3.28} $$

$$ \begin{aligned} {}_{VI}\mathfrak{C}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1) \equiv {} & [\![-\operatorname{grad}^1 \operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_1) \blacktriangle \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(2)}(\mathbf{x},\mathbf{x}_1)]\!] \\ & + \left[\!\!\left[\int_{\boldsymbol{\Sigma}_2} \operatorname{grad}^1 \boldsymbol{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) : [(\operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_2) \otimes \mathbf{n}(\boldsymbol{\sigma}_2)) \,\Delta\, \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\boldsymbol{\Sigma}_2 \right.\right. \\ & + \int_{\boldsymbol{\xi}_2} \operatorname{grad}^1 \boldsymbol{\Lambda}_{\mathbf{F}}(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) : [\operatorname{grad}^2 \operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_2) \,\Delta\, \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\boldsymbol{\xi}_2 \\ & \left.\left. - \int_{\boldsymbol{\xi}_2} \operatorname{grad} \dot{\boldsymbol{\Omega}}_H(\boldsymbol{\xi},\boldsymbol{\xi}_2) : [\operatorname{grad}^1 \operatorname{grad}^2 \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) \,\Delta\, \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)]\,\mathrm{d}\boldsymbol{\xi}_2 \right]\!\!\right] + \cdots \end{aligned} \tag{3.29} $$

and

$$ \begin{aligned} {}_{VI}\mathfrak{F}_2(\boldsymbol{\xi},\boldsymbol{\xi}_1) \equiv {} & [\![\operatorname{grad} \dot{\boldsymbol{\Omega}}_H(\boldsymbol{\xi},\boldsymbol{\xi}_1) : \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(2)}(\mathbf{x},\mathbf{x}_1)]\!] \\ & + \left[\!\!\left[\int_{\boldsymbol{\Sigma}_2} \Lambda_H(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1)\,(\operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_2) \otimes \mathbf{n}\,(\boldsymbol{\sigma}_2)) \,\Delta\, \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)\,\mathrm{d}\boldsymbol{\Sigma}_2 \right.\right. \\ & + \int_{\boldsymbol{\xi}_2} [\operatorname{grad} \dot{\boldsymbol{\Omega}}_H(\boldsymbol{\xi},\boldsymbol{\xi}_2) \otimes \operatorname{grad}^2 \dot{\boldsymbol{\Omega}}_H(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) \\ & \left.\left. - \Lambda_H(\boldsymbol{\xi}_2,\boldsymbol{\xi}_1) \operatorname{grad}^2 \operatorname{grad} \dot{\boldsymbol{\Omega}}_{\mathbf{F}}(\boldsymbol{\xi},\boldsymbol{\xi}_2)] \,\Delta \boldsymbol{\Psi}_{\boldsymbol{\Theta}}^{(3)}(\mathbf{x},\mathbf{x}_2,\mathbf{x}_1)\,\mathrm{d}\boldsymbol{\xi}_2 \right]\!\!\right] + \ldots \end{aligned} \tag{3.30} $$

We can now obtain the governing equation of the coupled mean fields, the mean displacement and the mean temperature, induced in the heterogeneous thermoelastic material by substituting the equations obtained above into eqn (3.3);

$$ \begin{aligned} \langle \rho \rangle \langle \ddot{\mathbf{u}}(\boldsymbol{\xi}) \rangle + \operatorname{div}\,[\langle \mathbf{C} \rangle : \operatorname{grad} \langle \mathbf{u}(\boldsymbol{\xi}) \rangle + \langle \boldsymbol{\Theta} \rangle \langle \theta(\boldsymbol{\xi}) \rangle] + {}_{\mathbf{u}}\chi(\boldsymbol{\xi}) &= -\mathbf{F}(\boldsymbol{\xi}) \\ \langle \boldsymbol{\lambda} \rangle : \operatorname{grad} \operatorname{grad} \langle \theta(\boldsymbol{\xi}) \rangle + \langle c \rangle \langle \dot{\theta}(\boldsymbol{\xi}) \rangle + {}_{\theta}\chi(\boldsymbol{\xi}) &= -H(\boldsymbol{\xi}), \end{aligned} \tag{3.31} $$

where the functional ${}_{\mathbf{u}}\chi(\boldsymbol{\xi})$ is defined as ${}_{K}\chi(\boldsymbol{\xi})$ in eqn (3.13) with the new functions ${}_{\mathbf{u}}\mathfrak{A}_i(\boldsymbol{\xi},\boldsymbol{\xi}_1)$, ${}_{\mathbf{u}}\mathfrak{B}_i(\boldsymbol{\xi},\boldsymbol{\xi}_1)$, ${}_{\mathbf{u}}\mathfrak{C}_i(\boldsymbol{\xi},\boldsymbol{\xi}_1)$, ${}_{\mathbf{u}}\mathfrak{D}_i(\boldsymbol{\xi},\boldsymbol{\xi}_1)$, ${}_{\mathbf{u}}\mathfrak{E}_i(\boldsymbol{\xi},\boldsymbol{\xi}_1)$ and ${}_{\mathbf{u}}\mathfrak{F}_i(\boldsymbol{\xi},\boldsymbol{\xi}_1)$: $(i = 1, 2)$ in place of ${}_{I}\mathfrak{A}_1(\boldsymbol{\xi},\boldsymbol{\xi}_1),\ldots$, etc. The tensor functions ${}_{\mathbf{u}}\mathfrak{A}_1,\ldots$, etc. are defined by the following formulae:

$$ \begin{aligned} {}_{\mathbf{u}}\mathfrak{A}_i &= {}_{I}\mathfrak{A}_i + \operatorname{div}\,[{}_{II}\mathfrak{A}_i + {}_{III}\mathfrak{A}_i] \\ \operatorname{div} {}_{\mathbf{u}}\mathfrak{B}_i &= {}_{I}\mathfrak{B}_i + \operatorname{div}\,[{}_{II}\mathfrak{B}_i + {}_{III}\mathfrak{B}_i] \\ \operatorname{div} {}_{\mathbf{u}}\mathfrak{C}_i &= {}_{I}\mathfrak{C}_i + \operatorname{div}\,[{}_{II}\mathfrak{C}_i + {}_{III}\mathfrak{C}_i] \\ {}_{\mathbf{u}}\mathfrak{D}_i &= {}_{I}\mathfrak{D}_i + \operatorname{div}\,[{}_{II}\mathfrak{D}_i + {}_{III}\mathfrak{D}_i] \\ {}_{\mathbf{u}}\mathfrak{E}_i &= {}_{I}\mathfrak{E}_i + \operatorname{div}\,[{}_{II}\mathfrak{E}_i + {}_{III}\mathfrak{E}_i] \\ {}_{\mathbf{u}}\mathfrak{F}_i &= {}_{I}\mathfrak{F}_i + \operatorname{div}\,[{}_{II}\mathfrak{F}_i + {}_{III}\mathfrak{F}_i];\ (i = 1, 2), \end{aligned} \tag{3.32} $$

The functional ${}_{\theta}\chi(\xi)$ is also determined by eqn (3.13) with the functions

$$
\begin{aligned}
{}_{\theta}\mathfrak{A}_i &= \sum_{K=IV}^{VI} {}_K\mathfrak{A}_i, \quad {}_{\theta}\mathfrak{B}_i = \sum_{K=IV}^{VI} {}_K\mathfrak{B}_i, \quad {}_{\theta}\mathfrak{C}_i = \sum_{K=IV}^{VI} {}_K\mathfrak{C}_i, \\
{}_{\theta}\mathfrak{D}_i &= \sum_{K=IV}^{VI} {}_K\mathfrak{D}_1, \quad \theta\mathfrak{E}_i = \sum_{K=IV}^{VI} {}_K\mathfrak{E}_i, \quad {}_{\theta}\mathfrak{F}_i = \sum_{K=IV}^{VI} {}_K\mathfrak{F}_i;
\end{aligned}
\qquad (i = 1, 2) \tag{3.33}
$$

in place of ${}_I\mathfrak{A}_1, \ldots$, etc.

It can be concluded from eqn (3.31) that the governing mean field equation of the heterogeneous thermoelastic media which are the assemblages of the linear thermoelastic solids exhibits the nonlocal property, though each constituent is governed by the equation with locality. The same conclusion was observed by Beran and McCoy[5] and Mazilu[24] for the heterogeneous linear elastic material, Beran and McCoy[25] for static electric field and Kröner[26] for the linear-dielectric heterogeneous field. We should note that the effect of the coupling of the mean fields in eqn (3.31) is much more remarkable than that in the deterministic eqn (2.1).

4. REDUCTION TO THE UNCOUPLED CASE

This section is concerned with the reduction of the general formulation in the preceding section to the uncoupled case. For the sake of simplicity, we restrict our discussion to the quasi-static case.

The problem reduces to the uncoupled one if the condition

$$\mathbf{\Theta}: \operatorname{grad} \dot{\mathbf{u}} = 0 \tag{4.1}$$

holds. Under this condition, the heat conduction eqn $(2.1)_2$ loses the dependence of the velocity $\mathbf{u}$. It is, therefore, clear that the correlation functions about the material constants ρ, $\mathbf{C}$ and $\mathbf{\Theta}$ do not appear in the governing equation of mean temperature field. This means that the functional ${}_{\theta}\chi(\xi)$ in eqn $(3{,}31)_2$ is represented by the terms of grad grad $\langle\theta\rangle$ and $\langle\dot{\theta}\rangle$ only. It is also obvious that it does not include the surface integrals since the heat conduction equation does not contain the operator div. From these considerations, we can conclude that the governing equation of mean temperature, eqn $(3.31)_2$, reduces to

$$
\begin{aligned}
\langle\boldsymbol{\lambda}\rangle: \operatorname{grad}\operatorname{grad}\langle\theta\rangle + \langle c\rangle\langle\dot{\theta}\rangle &+ \int_{\xi_1} [{}_{IV}\bar{\mathfrak{D}}_2(\xi, \xi_1) + {}_V\bar{\mathfrak{D}}_2(\xi, \xi_1)]: \operatorname{grad}^1 \operatorname{grad}^1 \langle\theta(\xi_1)\rangle\, d\xi_1 \\
&+ \int_{\xi_1} [{}_{IV}\bar{\mathfrak{E}}_2(\xi, \xi_1) + {}_V\bar{\mathfrak{E}}_2(\xi, \xi_1)]\langle\dot{\theta}(\xi_1)\rangle\, d\xi_1 = -H
\end{aligned} \tag{4.2}
$$

for the uncoupled case, where the tensor functions ${}_{IV}\bar{\mathfrak{D}}_2$, ${}_V\bar{\mathfrak{D}}_2$, ${}_{IV}\bar{\mathfrak{E}}_2$ and ${}_V\bar{\mathfrak{E}}_2$ are calculated by eliminating all the correlation functions for the material constants except $\boldsymbol{\lambda}$ and c, and taking account of eqn (2,17) in the expressions of ${}_{IV}\mathfrak{D}_2$, ${}_V\mathfrak{D}_2$, ${}_{IV}\mathfrak{E}_2$ and ${}_V\mathfrak{E}_2$ evaluated by eqns (3.23) and (3.25). The explicit formulae of the tensor functions ${}_{IV}\bar{\mathfrak{D}}_2$, etc. and the detailed considerations on eqn (4.2) were fully discussed by the same authors in Ref. [27].

For later use, we point out that θ' can be written as

$$\theta'(\xi) = \int_{\xi_1} [\check{G}_1(\xi, \xi_1): \operatorname{grad}^1 \operatorname{grad}^1 \langle\theta(\xi_1)\rangle + \check{G}_2(\xi, \xi_1)\langle\dot{\theta}(\xi_1)\rangle]\, d\xi_1 \tag{4.3}$$

together with

$$
\begin{aligned}
\check{G}_1(\xi, \xi_1) &\equiv [\![G(\xi, \xi_1)\,\boldsymbol{\lambda}'(\mathbf{x}_1)]\!] + \Big[\!\!\Big[(I-P)\int_{\xi_2} G(\xi, \xi_2)\,[\operatorname{grad}^2 \operatorname{grad}^2 G(\xi_2, \xi_1) \\
&\quad :(\boldsymbol{\lambda}'(\mathbf{x}_2)\otimes\boldsymbol{\lambda}'(\mathbf{x}_1)) + \dot{G}(\xi_2, \xi_1)\,(c'(\mathbf{x}_2)\,\boldsymbol{\lambda}'(\mathbf{x}_1))]\, d\xi_2\Big]\!\!\Big] + \ldots, \\
\check{G}_2(\xi, \xi_1) &\equiv [\![G(\xi, \xi_1)c'(\mathbf{x}_1)]\!] + \Big[\!\!\Big[(I-P)\int_{\xi_2} G(\xi, \xi_1)\,[\operatorname{grad}^2 \operatorname{grad}^2 G(\xi_2, \xi_1) \\
&\quad :(\boldsymbol{\lambda}'(\mathbf{x}_2)\,c'(\mathbf{x}_1)) + \dot{G}(\xi_2, \xi_1)\,(c'(\mathbf{x}_2)\,c'(\mathbf{x}_1))]\, d\xi_2\Big]\!\!\Big] + \ldots
\end{aligned} \tag{4.4}
$$

Equation (4.3) is obtained by the direct substitution of eqns (4.1) and (2.17) into eqn (3.11). It is worth noting that eqn (4.3) can be rewritten as

$$\theta'(\boldsymbol{\xi}) = \check{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) \langle \theta(\boldsymbol{\zeta}) \rangle \tag{4.5}$$

symbolically, where $\check{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta})$ represents the integro-differential operator determined by eqn (4.3). The $\boldsymbol{\zeta}$, which stands for the argument of the operator, means $(\mathbf{x}, t)$.

Below, we limit our attention to the equilibrium equation. Our purpose is to derive the explicit form of the equilibrium equation

$$\operatorname{div} [\langle \mathbf{C} \rangle : \langle \boldsymbol{\gamma} \rangle + \langle \boldsymbol{\Theta} \rangle \langle \theta \rangle] + {}_{\mathbf{u}}\boldsymbol{\chi}(\boldsymbol{\xi}) = -\mathbf{F} \tag{4.6}$$

and to consider the behavior of the materials governed by it. The $\boldsymbol{\gamma}$ in eqn (4.6) indicates the strain tensor defined by

$$\boldsymbol{\gamma} = \tfrac{1}{2}[\operatorname{grad} \mathbf{u} + (\operatorname{grad} \mathbf{u})^T]. \tag{4.7}$$

The equations for $\langle \mathbf{u} \rangle$ and $\mathbf{u}'$, eqns (3.3) and (3.4), now reduce to

$$\operatorname{div} [\langle \mathbf{C} \rangle : \langle \boldsymbol{\gamma} \rangle + \langle \boldsymbol{\Theta} \rangle \langle \theta \rangle + \langle \mathbf{C}' : \boldsymbol{\gamma}' \rangle + \tilde{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) \langle \theta(\boldsymbol{\zeta}) \rangle] + \mathbf{F} = \mathbf{0} \tag{4.8}$$

and

$$\operatorname{div} [\langle \mathbf{C} \rangle : \boldsymbol{\gamma}' + \mathbf{C}' : \langle \boldsymbol{\gamma} \rangle + (I - P)(\mathbf{C}' : \boldsymbol{\gamma}') + \bar{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) \langle \theta(\boldsymbol{\zeta}) \rangle] = 0, \tag{4.9}$$

where the following integro-differential operators are introduced:

$$\begin{aligned} \tilde{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) &\equiv \langle \boldsymbol{\Theta}'(\mathbf{x}) \check{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) \rangle, \\ \bar{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) &\equiv \langle \boldsymbol{\Theta}(\mathbf{x}) \rangle \check{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) + \boldsymbol{\Theta}'(\mathbf{x}) \check{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) - \langle \boldsymbol{\Theta}'(\mathbf{x}) \check{\Pi}(\boldsymbol{\xi}, \boldsymbol{\zeta}) \rangle + \boldsymbol{\Theta}'(\mathbf{x}) \mathrm{I}(\boldsymbol{\xi}, \boldsymbol{\zeta}). \end{aligned} \tag{4.10}$$

The operator $I(\boldsymbol{\xi}, \boldsymbol{\zeta})$ in the above equation means

$$\mathbf{A}(\boldsymbol{\xi}) = I(\boldsymbol{\xi}, \boldsymbol{\zeta}) \, \mathbf{A}(\boldsymbol{\zeta}) \tag{4.11}$$

for any arbitrary tensor $\mathbf{A}(\boldsymbol{\xi})$, while the juxtaposition $\mathbf{A}\check{\Pi}$ is defined as

$$\mathbf{A}\check{\Pi}\langle \theta \rangle \equiv \int_{\boldsymbol{\zeta}} [(\mathbf{A} \otimes \check{G}_1(\boldsymbol{\xi}, \boldsymbol{\zeta})) : \operatorname{grad} \operatorname{grad} \langle \theta(\boldsymbol{\zeta}) \rangle + \mathbf{A}\check{G}_2(\boldsymbol{\xi}, \boldsymbol{\zeta}) \langle \dot{\theta}(\boldsymbol{\zeta}) \rangle] \, \mathrm{d}\boldsymbol{\zeta}. \tag{4.12}$$

when $\langle \theta \rangle$ is operated. And then its average $\langle \mathbf{A}\check{\Pi} \rangle$ implies.

$$\langle \mathbf{A}\check{\Pi} \rangle \langle \theta \rangle \equiv \int_{\boldsymbol{\zeta}} [\langle \mathbf{A} \otimes \check{G}_1(\boldsymbol{\xi}, \boldsymbol{\zeta}) \rangle : \operatorname{grad} \operatorname{grad} \langle \theta(\boldsymbol{\zeta}) \rangle + \langle \mathbf{A} \, \check{G}_2(\boldsymbol{\xi}, \boldsymbol{\zeta}) \rangle \langle \dot{\theta}(\boldsymbol{\zeta}) \rangle] \, \mathrm{d}\boldsymbol{\zeta}. \tag{4.13}$$

Following the same type of disçussions in the preceding section, we can calculate $\langle \mathbf{C}' : \boldsymbol{\gamma}' \rangle$ by making use of the solution $\boldsymbol{\gamma}'$ of eqn (4.7). The result is

$$\begin{aligned} \langle \mathbf{C} : \boldsymbol{\gamma} \rangle = {} & \int_{\boldsymbol{\Sigma}_1} \mathfrak{B}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1) : \langle \boldsymbol{\gamma}(\boldsymbol{\xi}_1) \rangle \, \mathrm{d}\boldsymbol{\Sigma}_1 \\ & + \int_{\boldsymbol{\xi}_1} \mathfrak{B}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1) : \langle \boldsymbol{\gamma}(\boldsymbol{\xi}_1) \rangle \, \mathrm{d}\boldsymbol{\xi}_1 + \Pi_1(\boldsymbol{\xi}, \boldsymbol{\zeta}) \langle \theta(\boldsymbol{\zeta}) \rangle + \bar{\Pi}_2(\boldsymbol{\xi}, \boldsymbol{\zeta}) \langle \theta(\boldsymbol{\zeta}) \rangle, \end{aligned} \tag{4.14}$$

where the tensor functions $\mathfrak{B}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ and $\mathfrak{B}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ are given by

$$\begin{aligned} \mathfrak{B}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \equiv {} & [\![(\mathscr{G}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_1)) \boldsymbol{\Delta} \boldsymbol{\Psi}_{\mathrm{C}}^{(2)}(\mathbf{x}, \mathbf{x}_2)]\!] \\ & + \Bigg[\!\!\Bigg[\int_{\boldsymbol{\Sigma}_2} (\mathscr{G}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_2)) \boldsymbol{\Delta} [(\mathscr{G}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_1)) \boldsymbol{\Delta} \boldsymbol{\Psi}_{\mathrm{C}}^{(3)}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)] \, \mathrm{d}\boldsymbol{\Sigma}_2 \\ & - \int_{\boldsymbol{\xi}_2} \operatorname{grad}^2 \mathscr{G}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \boldsymbol{\Delta} [(\mathscr{G}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \otimes \mathbf{n}(\boldsymbol{\sigma}_1)) \boldsymbol{\Delta} \boldsymbol{\Psi}_{\mathrm{C}}^{(3)}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)] \, \mathrm{d}\boldsymbol{\xi}_2 \Bigg]\!\!\Bigg] + \cdots, \end{aligned}$$

$$\mathfrak{B}_2(\xi, \xi_1) \equiv [\![-\text{grad}^1\, \mathscr{G}(\xi, \xi_1) \blacktriangle \Psi_C^{(2)}(\mathbf{x}, \mathbf{x}_1)]\!]$$
$$+ [\![-\int_{\Sigma_2} (\mathscr{G}(\xi, \xi_2) \otimes \mathbf{n}(\sigma_2)) \Delta [\text{grad}^1\, \mathscr{G}(\xi_2, \xi_1) \Delta \Psi_C^{(3)}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)]\, \mathrm{d}\Sigma_2$$
$$+ \int_{\xi_2} \text{grad}^2\, \mathscr{G}(\xi, \xi_2)\, \Delta\, [\text{grad}^1\, \xi_2, \xi_1)\, \Delta\, \Psi_C^{(3)}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)]\, \mathrm{d}\xi_2]\!] + \cdots \tag{4.15}$$

In the above equations, the modified Green's tensor

$$\mathscr{G}_{ijk}(\xi, \xi_1) \equiv \tfrac{1}{2}[K_{ij,k}(\xi, \xi_1) + K_{kj,i}(\xi, \xi_1)] \tag{4.16}$$

is introduced. And the operators $\Pi_1(\xi, \zeta)$ and $\bar{\Pi}_2(\xi, \zeta)$ are defined by the following formulae:

$$\Pi_1(\xi, \zeta) \equiv [\![\int_{\Sigma_1} \mathscr{G}(\xi, \xi_1) \otimes \mathbf{n}(\sigma_1))\, \Delta\, \langle \mathbf{C}'(\mathbf{x})\, \bar{\Pi}(\xi_1, \zeta) \rangle\, \mathrm{d}\Sigma_1]\!]$$
$$+ [\![\int_{\Sigma_1} \Big[\int_{\Sigma_2} (\mathscr{G}(\xi, \xi_2) \otimes \mathbf{n}(\sigma_2) \blacktriangle \{ (\mathscr{G}(\xi_2, \xi_1) \otimes \mathbf{n}(\sigma_2))\, \blacktriangle\, \langle (\mathbf{C}'(\mathbf{x}) \otimes \mathbf{C}'(\mathbf{x}_2))\, \check{\Pi}\, (\xi_1, \zeta) \rangle \}\, \mathrm{d}\Sigma_2$$
$$- \int_{\xi_2} \text{grad}^2\, \mathscr{G}(\xi, \xi_2)\, \blacktriangle\, \{ (\mathscr{G}(\xi_2, \xi_1) \otimes \mathbf{n}(\sigma_1))\, \blacktriangle\, \langle (\mathbf{C}'(\mathbf{x}) \otimes \mathbf{C}'(\mathbf{x}_2))\, \bar{\Pi}(\xi, \zeta) \rangle \}\, \mathrm{d}\xi_2 \Big]\, \mathrm{d}\Sigma_1]\!] + \ldots, \tag{4.17}$$

$$\bar{\Pi}_2(\xi, \zeta) \equiv [\![-\int_{\xi_1} \text{grad}^1 \mathscr{G}(\xi, \xi_1) \Delta \langle \mathbf{C}'(\mathbf{x}) \bar{\Pi}(\xi, \zeta)]\, \langle \boldsymbol{\theta}(\zeta) \rangle\, \mathrm{d}\xi_1]\!]$$
$$+ [\![\int_{\xi_1} \Big[-\int_{\Sigma_2} (\mathscr{G}(\xi, \xi_2) \otimes \mathbf{n}(\sigma_2)) \blacktriangle \{ \text{grad}^1\, \mathscr{G}(\xi_2, \xi_1) \blacktriangle \langle (\mathbf{C}'(\mathbf{x}) \otimes \mathbf{C}'(\mathbf{x}_2)) \bar{\Pi}(\xi, \zeta) \rangle \}\, \mathrm{d}\Sigma_2$$
$$+ \int_{\xi_2} \text{grad}^2\, \mathscr{G}(\xi, \xi_2) \blacktriangle \{ \text{grad}^1\, \mathscr{G}(\xi_2, \xi_1) \blacktriangle\, \langle (\mathbf{C}'(\mathbf{x}) \otimes \mathbf{C}'(\mathbf{x}_2))\, \bar{\Pi}(\xi, \zeta) \rangle \}\, \mathrm{d}\xi_2 \Big]\, \mathrm{d}\xi_1]\!] + \cdots$$

The tensor functions $\mathfrak{B}_1(\xi, \xi_1)$ and $\mathfrak{B}_2(\xi, \xi_1)$ defined by eqn (4.15) correspond to the ones which are determined by retaining the correlation functions of **C** only in the expressions of ${}_{II}\mathfrak{B}_1(\xi, \xi_1)$ and ${}_{II}\mathfrak{B}_2(\xi, \xi_1)$, eqns (3.17) and (3.18), and bearing in mind of eqn (2.17).

Substitution of eqn (4.14) into eqn (4.8) leads the final governing equation of mean fields

$$\text{div}\, [\langle \mathbf{C} \rangle : \langle \boldsymbol{\gamma} \rangle + \langle \boldsymbol{\Theta} \rangle \langle \boldsymbol{\theta} \rangle + \int_{\Sigma_1} \mathfrak{B}_1(\xi, \xi_1) : \langle \boldsymbol{\gamma}(\xi_1) \rangle\, \mathrm{d}\Sigma_1 + \int_{\xi_1} \mathfrak{B}_2(\xi, \xi_1) : \langle \boldsymbol{\gamma}(\xi_1) \rangle\, \mathrm{d}\xi_1$$
$$+ \Pi_1(\xi, \zeta)\, \langle \boldsymbol{\theta}(\zeta) \rangle + \Pi_2(\xi, \zeta) \rangle] + \mathbf{F} = \mathbf{0} \tag{4.18}$$

where

$$\Pi_2(\xi, \zeta) \equiv \bar{\Pi}_2(\xi, \zeta) + \tilde{\Pi}(\xi, \zeta). \tag{4.19}$$

Equations (4.18) and (4.2) constitute a set of the field equations for the uncoupled mean fields $\langle \boldsymbol{\gamma} \rangle$ and $\langle \theta \rangle$ induced in the heterogeneous thermoelastic materials. We emphasize that the term $\Pi_1(\xi, \zeta)\, \langle \theta(\zeta) \rangle + \Pi_2(\xi, \zeta)\, \langle \theta(\zeta) \rangle$ in eqn (4.18) may be regarded as a function of ξ which is independent of the mechanical history of the material since the field $\langle \theta \rangle$ is determined independently of $\langle \boldsymbol{\gamma} \rangle$, and Green's tensor $\mathscr{G}(\xi, \xi_1)$ is a prescribed function.

We postulate here the following assumptions in order to take account of the behavior of the actual materials. The tensor functions $\mathfrak{B}_1(\xi, \xi_1)$ and $\mathfrak{B}_2(\xi, \xi_1)$ are assumed to be negligibly small when the distance $|\mathbf{x} - \mathbf{x}_1|$ is greater than the characteristic length l associated with the statistical variations of **C**. This means that the only values evaluated at the points $\mathbf{x}_1$ which enter the region $|\mathbf{x} - \mathbf{x}_1| \leqq l$ contribute to the integration. We further postulate that the variation of the mean field is so slow that $\langle \boldsymbol{\gamma}(\mathbf{x}_1, t_1) \rangle$ can be expanded in Taylor series about the point $\mathbf{x}$, i.e.

$$\langle \boldsymbol{\gamma}(\mathbf{x}_1, t_1) \rangle = \langle \boldsymbol{\gamma}(\mathbf{x}, t) \rangle + (\mathbf{x}_1 - \mathbf{x}) . \text{grad}\, \langle \boldsymbol{\gamma}(\mathbf{x}, t_1) \rangle + \ldots$$

Under these assumptions, eqn (4.18) is reduced to

$$\text{div}\, \langle \boldsymbol{\sigma} \rangle + \mathbf{F} = \mathbf{0},$$

$$\langle \boldsymbol{\sigma}(\mathbf{x}, t)\rangle = \langle \mathbf{C}\rangle : \langle \boldsymbol{\gamma}(\mathbf{x}, t)\rangle + \langle \boldsymbol{\Theta}\rangle \langle \theta(\mathbf{x}, t)\rangle + \boldsymbol{\pi}_1(\mathbf{x}, t, \boldsymbol{\zeta}) \langle \theta(\boldsymbol{\zeta})\rangle + \boldsymbol{\pi}_2(\mathbf{x}, t, \boldsymbol{\zeta})\langle \theta(\boldsymbol{\zeta})\rangle$$
$$+ \int_{t_1} \mathcal{B}^{(1)}(\mathbf{x}, t, t_1) : \langle \boldsymbol{\gamma}(\mathbf{x}, t_1)\rangle \, \mathrm{d}t_1 + \int_{t_1} \mathcal{B}^{(2)}(\mathbf{x}, t, t_1) \blacksquare \operatorname{grad} \langle \boldsymbol{\gamma}(\mathbf{x}, t_1)\rangle \, \mathrm{d}t_1 + \cdots, \tag{4.20}$$

where $\langle \boldsymbol{\sigma}\rangle$ indicates the mean stress field, while the tensor functions $\mathcal{B}^{(n)}(\mathbf{x}, t, t_1)$; $(n = 1, 2, \ldots)$ are defined by

$$\mathcal{B}^{(1)}(\mathbf{x}, t, t_1) \equiv \int_{\mathcal{S}_1} \mathfrak{B}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \, \mathrm{d}\boldsymbol{\sigma}_1 + \int_{\mathcal{V}_1} \mathfrak{B}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \, \mathrm{d}\mathbf{x}_1$$
$$\mathcal{B}^{(2)}(\mathbf{x}, t, t_1) \equiv \int_{\mathcal{S}_1} \mathfrak{G}_1(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \otimes (\mathbf{x}_1 - \mathbf{x}) \, \mathrm{d}\boldsymbol{\sigma}_1 + \int_{\mathcal{V}_1} \mathfrak{B}_2(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \otimes (\mathbf{x}_1 - \mathbf{x}) \, \mathrm{d}\mathbf{x}_1 \ldots. \tag{4.21}$$

Though $\boldsymbol{\sigma}$ in $\langle \boldsymbol{\sigma}\rangle$ is the same symbol as the coordinates on the body surface $\boldsymbol{\sigma}_i$, no confusion will come frum the usage. The notations $\mathcal{S}_i$ and $\mathcal{V}_i$ in the above equation mean the restricted regions

$$\mathcal{S}_i \equiv \{\boldsymbol{\sigma}_i \,|\, |\boldsymbol{\sigma} - \boldsymbol{\sigma}_i| \leq l\}$$
$$\mathcal{V}_i \equiv \{\mathbf{x}_i \,|\, |\mathbf{x} - \mathbf{x}_i| \leq l\}.$$

The operators $\boldsymbol{\pi}_1(\mathbf{x}, t, \boldsymbol{\zeta})$ and $\boldsymbol{\pi}_2(\mathbf{x}, t, \boldsymbol{\zeta})$ are defined as ones which are calculated by restricting the spatial integrals with respect to $\boldsymbol{\sigma}_i$ and $\mathbf{x}_i$ within $\mathcal{S}_i$ and $\mathcal{V}_i$ in the definitions of $\Pi_1(\boldsymbol{\xi}, \boldsymbol{\zeta})$ and $\Gamma_2(\boldsymbol{\xi}, \boldsymbol{\zeta})$, eqns (4.17) and (4.19).

Equation $(4.20)_2$ may be regarded as the constitutive equation for the mean fields of the heterogeneous thermoelastic media. However, we should note that the equation includes the effects of the shape of the material and the boundary conditions through Green's function. This shows that, as Beran and McCoy [5] pointed out, eqn $(4.20)_2$ does not correspond to the constitutive equation employed in the rational continuum mechanics [28].

The further discussions on the relation $(4.20)_2$ will be made in a sequel paper by the same authors [29]. In that paper they will reveal that the integral on the mean strain are no more than a pretense.

5. CONCLUDING REMARKS

A formulation was proposed on the thermomechanical behavior of the heterogeneous linear thermoelastic media. The governing equation of the coupled mean fields induced in the material was derived. It was shown that the governing equation exhibited the nonlocality though each constituent was governed by the local law.

The reduced uncoupled case was examined in detail. The relation of the mean stress, the mean strain and the mean temperature was derived when the acceleration term was disregarded.

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APPENDIX

It follows from eqn (3.10) that

$$\langle \mathbf{C}' : \operatorname{grad} \mathbf{u}'^{(2)} \rangle = \int_{\xi_1} \operatorname{grad} \boldsymbol{\Omega}_F(\xi, \xi_1) \blacksquare \operatorname{div}^1 \left[\int_{\xi_2} \operatorname{grad}^1 \boldsymbol{\Omega}_F(\xi_1, \xi_2) \blacksquare \operatorname{div}^2 (\boldsymbol{\Psi}_C^{(3)}(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2) : \operatorname{grad}^2 \langle \mathbf{u}(\xi_2) \rangle)\, d\xi_2 \right] d\xi_1$$

$$= \int_{\xi_1} \operatorname{grad} \boldsymbol{\Omega}_F(\xi, \xi_1) \blacksquare \operatorname{div}^1 \mathscr{A}(\xi, \xi_1)\, d\xi_1 \tag{A1}$$

where the following tensor is defined:

$$\mathscr{A}(\xi, \xi_1) \equiv \int_{\xi_2} \operatorname{grad}^1 \boldsymbol{\Omega}_F(\xi_1, \xi_2) \blacksquare \operatorname{div}^2 (\boldsymbol{\Psi}_C^{(3)}(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2) : \operatorname{grad}^2 \langle \mathbf{u}(\xi_2) \rangle)\, d\xi_2. \tag{A2}$$

The symbol $\blacksquare$ in the above equations stands for an operator of the summation over the repeated three indices,

$$[\mathbf{A} \blacksquare \mathbf{B}]_{\ldots} \equiv A_{\ldots ijk} B_{.ikj}$$

Use of Gauss' theorem leads eqn (A1) to

$$\langle \mathbf{C}' : \operatorname{grad} \mathbf{u}'^{(2)} \rangle \equiv \int_{\xi_1} [\operatorname{div}^1(\operatorname{grad} \boldsymbol{\Omega}_F(\xi, \xi_1) \square \mathscr{A}(\xi, \xi_1)) - \operatorname{grad}^1 \operatorname{grad}^1 \boldsymbol{\Omega}_F(\xi, \xi_1) \blacktriangle \mathscr{A}(\xi, \xi_1)]\, d\xi_1$$

$$= \int_{\Sigma_1} \operatorname{grad} \boldsymbol{\Omega}_F(\xi, \xi_1) \blacksquare (\mathscr{A}(\xi, \xi_1) \cdot \mathbf{n}(\sigma_1))\, d\Sigma_1 - \int_{\xi_1} \operatorname{grad}^1 \operatorname{grad} \boldsymbol{\Omega}_F(\xi, \xi_1) \blacktriangle \mathscr{A}(\xi, \xi_1)\, d\xi_1, \tag{A3}$$

where we have introduced the following operators:

$$[\mathbf{A} \square \mathbf{B}]_{..l} \equiv A_{..ijk} B_{..ikjl},$$
$$[\mathbf{A} \blacktriangle \mathbf{B}]_{.} \equiv A_{..ijkl} B_{...ikjl}$$

Equation (A2) is also rewritten by using Gauss' theorem to

$$\mathscr{A}(\xi, \xi_1) = \int_{\Sigma_2} \operatorname{grad}^1 \boldsymbol{\Omega}_F(\xi_1, \xi_2) \blacksquare [\boldsymbol{\Psi}_C^{(3)}(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2) : \operatorname{grad}^2 \langle \mathbf{u}(\xi_2) \rangle) \cdot \mathbf{n}(\sigma_2)]\, d\Sigma_2$$

$$- \int_{\xi_2} \operatorname{grad}^2 \operatorname{grad}^1 \boldsymbol{\Omega}_F(\xi_1, \xi_2) \blacktriangle [\boldsymbol{\Psi}_C^{(3)}(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2) : \operatorname{grad}^2 \langle \mathbf{u}(\xi_2) \rangle]\, d\xi_2. \tag{A4}$$

Substitution of eqn (A4) into eqn (A3) and rearrangement of terms give the final form of $\langle \mathbf{C}' : \operatorname{grad} \mathbf{u}'^{(2)} \rangle$,

$$\langle \mathbf{C}' : \operatorname{grad} \mathbf{u}'^{(2)} \rangle = \int_{\Sigma_1} {}_{II}\mathfrak{B}_1^{(2)}(\xi, \xi_1) : \operatorname{grad}^1 \langle \mathbf{u}(\xi_1) \rangle\, d\Sigma_1 + \int_{\xi_1} {}_{II}\mathfrak{B}_2^{(2)}(\xi, \xi_1) : \operatorname{grad}^1 \langle \mathbf{u}(\xi_1) \rangle\, d\xi_1, \tag{A5}$$

where the tensors ${}_{II}\mathfrak{B}_1^{(2)}(\xi, \xi)$ and ${}_{II}\mathfrak{B}_2^{(2)}(\xi, \xi_1)$ are defined by

$${}_{II}\mathfrak{B}_1^{(2)}(\xi, \xi_1) \equiv \int_{\Sigma_2} (\operatorname{grad} \boldsymbol{\Omega}_F(\xi, \xi_2) \otimes \mathbf{n}(\sigma_2)) \triangle [(\operatorname{grad}^2 \boldsymbol{\Omega}_F(\xi_2, \xi_1) \otimes \mathbf{n}(\sigma_1)) \triangle \boldsymbol{\Psi}_C^{(3)}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)]\, d\Sigma_2 \tag{A6}$$

$$- \int_{\xi_2} \operatorname{grad}^2 \operatorname{grad} \boldsymbol{\Omega}_F(\xi, \xi_2) \triangle [(\operatorname{grad}^2 \boldsymbol{\Omega}_F(\xi_2, \xi_1) \otimes \mathbf{n}(\sigma_1)) \triangle \boldsymbol{\Psi}_C^{(3)}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)]\, d\xi_2$$

and

$$ {}_{II}\mathfrak{B}_2^{(2)}(\boldsymbol{\xi}, \boldsymbol{\xi}_1) \equiv - \int_{\Sigma_2} (\text{grad}\, \boldsymbol{\Omega}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \otimes \mathbf{n}(\boldsymbol{\sigma}_2)) \triangle [\text{grad}^1\, \text{grad}^2\, \boldsymbol{\Omega}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \triangle \boldsymbol{\Psi}_{\mathrm{C}}^{(3)}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)]\, \mathrm{d}\boldsymbol{\Sigma}_2 $$

$$ + \int_{\boldsymbol{\xi}_2} \text{grad}^2\, \text{grad}\, \boldsymbol{\Omega}_{\mathbf{F}}(\boldsymbol{\xi}, \boldsymbol{\xi}_2) \triangle [\text{grad}^1\, \text{grad}^2\, \boldsymbol{\Omega}_{\mathbf{F}}(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1) \triangle \boldsymbol{\Psi}_{\mathrm{C}}^{(3)}(\mathbf{x}, \mathbf{x}_2, \mathbf{x}_1)]\, \mathrm{d}\boldsymbol{\xi}_2. \tag{A7} $$

Comparison of eqn (A6) and (A7) with eqns (3.18) and (3.19) reveals that the tensors ${}_{II}\mathfrak{B}_1{}^{(2)}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ and ${}_{II}\mathfrak{B}_2{}^{(2)}(\boldsymbol{\xi}, \boldsymbol{\xi}_1)$ correspond to the second term is the infinite series.